Physics 216, Final Exam
January 10, 2013, 2 hours

Solve ten problems out of twelve

1. Check the divergence theorem for the vector $\overrightarrow{V} = r^2 \cos\theta \widehat{r} + r^2 \cos\phi \widehat{\theta} - r^2 \cos\theta \sin\phi \widehat{\phi}$ using as your volume a hemisphere of radius R. Take the flat side in the $x - y$ plane.

2. Expand the function $f(x) = x^2$, $0 < x < 2\pi$ in a Fourier series if the period is 2π.

3. Find the eigenvalues and normalized eigenvectors of the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}$$

4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an orthogonal matrix satisfying $A^T A = I$ where I is the identity matrix. Find the most general solution for a, b, c, d satisfying the orthogonality condition. Compare with the rotation matrix in two dimensions.

5. Let $\{e_1, e_2, e_3\}$ be a basis of a three dimensional vector space V_3 and T an operator that transforms this basis to the vectors $\{e'_1, e'_2, e'_3\}$ where $e'_1 = T(e_1) = e_1 + e_3$, $e'_2 = T(e_2) = 2e_1 + e_2$, $e'_3 = T(e_3) = 3e_2 + e_3$. Find the matrix representation of the operator T and determine whether the vectors $\{e'_1, e'_2, e'_3\}$ are independent.

6. Find the Fourier transform of the function $f(x) = \begin{cases} 1 - |x|, & -a \le x \le a \\ 0, & \text{otherwise} \end{cases}$.

7. Verify the delta function expansion

$$\delta(1 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{2} P_n(x)$$

8. Find the analytic function $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = x^3 - 3xy^2$.

9. Let $\{Q_n, \quad n = 0, \cdots, \infty\}$ be an orthogonal set of Sturm-Liouville polynomials with respect to the weight function $w(x)$

$$\int_{-\infty}^{\infty} w(x) Q_n(x) Q_m(x)\, dx = 0, \qquad n \neq m$$

where $Q_n\ (x)$ satisfies the differential equation $\frac{1}{w}\frac{d}{dx}\left(w\alpha(x)\frac{dQ_n}{dx}\right) + \lambda_n Q_n = 0$. Show that $\left\{Q'_n = \frac{dQ_n}{dx}, \quad n = 1, \cdots, \infty\right\}$ satisfy the property

$$\int_{-\infty}^{\infty} w(x)\, \alpha(x)\, Q'_n(x)\, Q'_m(x)\, dx = 0, \qquad n \neq m$$

Hint: Integrate by parts only on one of the $\frac{dQ_n}{dx}$ and use the defining equation for Q_n.

10. Expand

$$f(z) = \frac{z^3 - 2z^2 + 1}{(z-3)^2 (z^2 + 3)}$$

in a Laurent series about $z = 3$ and $z = \pm i\sqrt{3}$.

11. Using complex integration over the unit circle show that

$$\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \frac{2\pi}{(a^2 - b^2)^{\frac{1}{2}}}, \qquad a > |b|$$

12. Find the residues of the function

$$f(z) = \frac{e^z}{z^2 (z^2 + 9)}$$

about the isolated singular points.